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SHAPE PRESERVING WIDTHS OF SOBOLEV-TYPE CLASSES OF s-MONOTONE FUNCTIONS ON A FINITE INTERVAL

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ABSTRACT

Let I be a finite interval and $r \in \mathbb{N}$. Denote by $\Delta_+^s L_q$ the subset of all functions $y \in L_q$ such that the s-difference $\Delta_\tau^s y(\cdot)$ is nonnegative on I, $\forall \tau > 0$. Further, denote by $\Delta_+^s W_p^r$ the class of functions x on I with the seminorm $\|x^{(r)}\|_{L_p} \le 1$, such that $\Delta_\tau^s x \ge 0$, $\tau > 0$. For $s = 3, \dots, r+1$, we obtain two-sided estimates of the shape preserving widths

$$d_n \left(\Delta_+^s W_p^r, \Delta_+^s L_q\right)_{L_q} := \inf_{M^n \in \mathcal{M}^n} \sup_{x \in \Delta_+^s W_p^r} \inf_{y \in M^n \cap \Delta_+^s L_q} \|x - y\|_{L_q},$$

where $\mathcal{M}^n$ is the set of all linear manifolds M^n in L_q, $\dim M^n \le n$, such that $M^n \cap \Delta_+^s L_q \neq \emptyset$.

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1. Introduction and statement of the main results

Let X be a real linear space of vectors x with a norm $\|x\|_X$, $W \subset X$, $W \neq \emptyset$ and $V \subset X$, $V \neq \emptyset$. Let L^n be a subspace in X of dimension $\dim L^n \leq n$, $n \geq 0$ and $M^n = M^n(z) := z + L^n$ be a shift of the subspace L^n by an arbitrary vector $z \in X$. If $M^n \cap V \neq \emptyset$, then we denote by

$$E(x, M^n \cap V)_X := \inf_{y \in M^n \cap V} \|x - y\|_X$$

the best approximation of the vector $x \in X$ by $M^n \cap V$, and by

$$E(W, M^n \cap V)_X := \sup_{x \in W} E(x, M^n \cap V)_X$$

the deviation of the set W from $M^n \cap V$.

Let $\mathcal{M}^n = \mathcal{M}^n(X, V)$ be the set of all linear manifolds M^n, $\dim M^n \leq n$ such that $M^n \cap V \neq \emptyset$. The quantity

$$d_n(W, V)_X := \inf_{M^n \in \mathcal{M}^n} E(W, M^n \cap V)_X, \quad n \geq 0$$

is called the relative n-width of W with the constraint V in X. These widths were introduced by the first author in [1].

Evidently, if $V = X$, then the relative n-width $d_n(W, V)_X$ coincides with the Kolmogorov n-width $d_n(W)_X$. Clearly, $d_n(W, V)_X \geq d_n(W)_X$.

Let I be a finite interval in $\mathbb{R}$, and let $r \in \mathbb{N}$. We will take $I = [-1, 1]$ as the generic interval and we will omit referring to it in the notation whenever the interval we deal with is I; for instance, we write $\|\cdot\|_{L_p}$ for $\|\cdot\|_{L_p(I)}$. We will use other intervals and the whole real line $\mathbb{R}$ and we will make clear in the notation whenever we deal with them. For $1 \leq p \leq \infty$, we denote

$$W_p^r := W_p^r(I) := \{x : I \to \mathbb{R} \mid x^{(r-1)} \in AC_{loc}(I), \|x^{(r)}\|_{L_p} \leq 1\},$$

where $AC_{loc}(I)$ is the collection of all functions defined on I which are absolutely continuous in every closed subinterval of $(-1, 1)$. Let

$$\Delta_\tau^s x(t) := \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} x(t + k\tau), \quad \{t, t + s\tau\} \subset I, \quad s = 0, 1, \ldots,$$

be the s-th difference of the function x, with step $\tau > 0$, and denote by$\Delta_+^s W_p^r = \Delta_+^s W_p^r(I)$, $s = 0, 1, \ldots$, the subclass of functions $x \in W_p^r$ for which $\Delta_\tau^s x(t) \geq 0$, for all $\tau > 0$ such that $[t, t+s\tau] \subseteq I$. By $\Delta_+^s L_q = \Delta_+^s L_q(I)$ we denote the subclass of all functions $y \in L_q(I)$ such that $\Delta_\tau^s y(t) \geq 0$, $\tau > 0$. In recent years, shape preserving approximation has become a central subject especially in applications. This is due to the fact that in CAGD and especially in questions of design, shape preservation is one of the main considerations. Our results below show what one may expect to achieve and what is beyond the reach of any approximation process which involves approximation from linear n-dimensional manifolds, when we preserve the shape of the approximants.

In this paper we prove the following

THEOREM: *Let $r \in \mathbb{N}$, $s \in \mathbb{N}$ and $1 \le p, q \le \infty$. For $3 \le s \le r$, we have*

$$d_n\left(\Delta_+^s W_p^r, \Delta_+^s L_q\right)_{L_q} \asymp n^{-r+s+1/p-3}, \quad n \ge r. \tag{1.1}$$

Also, if $s = r+1$, $r \ge 2$, then

$$d_n\left(\Delta_+^{r+1} W_p^r, \Delta_+^{r+1} L_q\right)_{L_q} \asymp n^{-2}, \quad n \ge r. \tag{1.2}$$

Remarks: (i) Note that the asymptotic relations are independent of q.

(ii) The upper bounds are achieved by piecewise polynomials of degree $\asymp r$, with n knots, that are elements of $\Delta_+^s C^{s-2}$. For $3 \le s \le r$, the knots are equidistant; however, if $s = r+1$, $r \ge 2$, then we are unable to guarantee that, and in our construction the knots are not equidistant.

(iii) It is worthwhile noting that as a byproduct we may conclude that the lower bound in (1.1) with $s = r > 3$ excludes the possibility of Jackson-type estimates involving the fourth modulus of smoothness of x evaluated at $1/n$, in s-monotone approximation of x, by piecewise polynomials or splines with n equidistant knots and thus also not by polynomials of degree $\le n$. Moreover, it even excludes Jackson-type estimates involving the generally bigger $Cn^{-3}\omega(x^{(3)}, n^{-1})_p$.

Recall that up until now we knew that Shvedov [6] had shown that Jackson-type estimates of s-monotone approximation of an s-monotone x, by polynomials of degree $\le n$, cannot be had with $C\omega_{s+2}(x, n^{-1})_p$. Thus the above is somewhat unexpected in view of what seemed like a pattern that we have Jackson-type estimates involving $C\omega_2(x, n^{-1})_p$ for monotone approximation, and by Shvedov [6], it is impossible to have such estimates with $\omega_3(x, n^{-1})_p$; and we have Jackson-type estimates for convex approximation involving $\omega_3(x, n^{-1})_p$, while again by Shvedov [6], it is impossible to have such estimates with $\omega_4(x, n^{-1})_p$.

It is interesting to compare the above asymptotic relations with earlier results. Surprisingly, they are quite different. For instance, for $s = 1, 2$, these relations, in general, do depend on q as we have shown in [5], namely, THEOREM KL1: *Let $s = 1,2$, $s \le r \in \mathbb{N}$, and $1 \le p,q \le \infty$, be such that $r - 1/p + 1/q > 0$. Then*

$$d_n(\Delta_+^s W_p^r, \Delta_+^s L_q)_{L_q} \asymp n^{-r+(1/p-1/q)_+}, \quad n \ge r.$$

If, on the other hand, $s = r + 1 = 2$, then

$$d_n(\Delta_+^2 W_p^1, \Delta_+^2 L_q)_{L_q} \asymp n^{-1-1/q}, \quad n \ge 1.$$

It is also worthwhile to see what kind of asymptotic relations are known for the unconstrained widths. In this case we have shown [3]

THEOREM KL2: *Let $r \in \mathbb{N}$ and $1 \le p,q \le \infty$ be such that $r - 1/p + 1/q > 0$. If, $(r,p) \ne (1,1)$, and if $(r,p) = (1,1)$ and $1 \le q \le 2$, then for each $s = 0,1,\dots,r$,*

$$d_n(\Delta_+^s W_p^r)_{L_q} \asymp n^{-r+(\max\{1/p,1/2\}-\max\{1/q,1/2\})_+}, \quad n \ge r.$$

If, on the other hand, $(r,p) = (1,1)$ and $2 < q < \infty$, then for $s = 0,1$,

$$c_1 n^{-\frac{1}{2}} \le d_n(\Delta_+^s W_1^1)_{L_q} \le c_2 n^{-1/2}\big(\log(n+1)\big)^{3/2}, \quad n \ge 1,$$

where $c_1 > 0$ and c_2 do not depend on n. Furthermore,

$$d_n(\Delta_+^{r+1} W_p^r)_{L_q} \asymp n^{-r-\max\{1/q,1/2\}}, \quad n > r.$$

2. Some auxiliary lemmas

In order to prove our theorem, we need a few lemmas. The first was proved by the authors in [5, Lemma 1].

LEMMA A: *Let J be a finite interval, and let $\{t_i\}_{i=1}^r$ be a collection of $r \in \mathbb{N}$ disjoint points in J. Set $\delta_1 := 1$ and $\delta_r := \min\{|t_i - t_j|, i \ne j\}$ for $r > 1$. Then for any function x such that $x^{(r)} \in L_1(J)$,*

$$\|x\|_{L_\infty(J)} \le \frac{r}{(r-1)!}\Big(\frac{|J|}{\delta_r}\Big)^{\frac{r(r-1)}{2}}\Big(\max_{1\le i\le r}|x(t_i)| + \frac{|J|^{r-1}}{(r-1)!}\|x^{(r)}\|_{L_1(J)}\Big).$$

A similar result was proved by the authors in [4, Lemma 1].

LEMMA B: *Let $J = [a,b]$ and $m \in \mathbb{N}$, and set $t_i = t_{m,i} := a + im^{-1}|J|$, $i = 0,\dots,m$. Then for every function x such that $x'' \in L_\infty(J)$,*

$$\|x'\|_{L_\infty(J)} \le 2m|J|^{-1}\max_{0\le i\le m}|x(t_i)| + \frac{1}{4}m^{-1}|J|\|x''\|_{L_\infty(J)}.$$

Next we need a result concerning Jackson-type estimates of the simultaneous approximation of 3-monotone function by 3-monotone quadratic splines with equidistant knots (see [4]).

LEMMA C: *Let $J = [a,b]$ and $x \in \Delta_+^3 W_p^3(J)$, $1 \le p \le \infty$. For $m \in \mathbb{N}$, let $t_i = t_{m,i} := a + im^{-1}|J|$, $i = 0,\dots,m$. Then there exists a quadratic spline $\sigma_{2,m}(x;\cdot)$ with knots t_i, $i = 0,1,\dots,m$, such that*

$$x''(t_{i-1}) \le \sigma_{2,m}''(x;t) \le x''(t_i), \quad t \in (t_{i-1}, t_i), \quad i = 1,\dots,m,$$

and

$$\|x(\cdot) - \sigma_{2,m}(x;\cdot)\|_{L_\infty(J)} \le \frac{3}{2} m^{-3+1/p}|J|^{3-1/p},$$
$$\|x'(\cdot) - \sigma_{2,m}'(x;\cdot)\|_{L_\infty(J)} \le \frac{7}{2} m^{-2+1/p}|J|^{2-1/p},$$
$$\|x''(\cdot) - \sigma_{2,m}''(x;\cdot)\|_{L_\infty(J)} \le m^{-1+1/p}|J|^{1-1/p}.$$

Next we have

LEMMA 1: *Let $J = [a,b]$ and $x \in C^s(J)$, $s \in \mathbb{N}$. If $\min_{t\in J}|x^{(s)}(t)| \ge A > 0$, then there exists a subinterval $J_s \subseteq J$, of length $|J_s| = 4^{-s}|J|$, such that $\min_{t\in J_s}|x(t)| \ge 2^{-s(s+1)}|J|^s A$.*

Proof: Denote $c := \frac{1}{2}(a+b)$, and assume $x^{(s-1)}(c) \ge 0$. Then from

$$x^{(s-1)}(t) = x^{(s-1)}(c) + \int_c^t x^{(s)}(\tau)d\tau, \quad t \in J,$$

it follows that if $x^{(s)}(t) \ge A$, $t \in J$, then $x^{(s-1)}(t) \ge A(t-c)$, $t \in [c,b]$. Hence for $J_1 := [\frac{1}{2}(c+b), b]$, which is of length $|J|/4$, we have $x^{(s-1)}(t) \ge 4^{-1}|J|A$. If, on the other hand, $x^{(s)}(t) \le -A$, $t \in J$, then $-x^{(s-1)}(t) \le -A(c-t)$, $t \in [a,c]$. Hence we have $-x^{(s-1)}(t) \le -4^{-1}|J|A$, in the interval $J_1 := [a, \frac{1}{2}(a+c)]$. The proof is similar if $x^{(s-1)}(c) \le 0$. Thus, in all cases we have established the existence of the interval J_1, $|J_1| \ge |J|/4$, such that $|x^{(s-1)}(t)| \ge 4^{-1}|J|A$, $t \in J_1$. Now the rest of the proof readily follows by induction. ∎

We apply Lemma 1 to prove

LEMMA 2: *For $s \in \mathbb{N}$, denote $\chi_s(t) := \frac{1}{s!}t_+^s$, $t \in \mathbb{R}$, and for $s > 1$ let $\xi \in C^s[-a,a]$, $a > 0$, be such that $\xi^{(s)}$ is nondecreasing and $0 \le \xi^{(s)} \le 1$ in $[-a,a]$. Then, if*

$$\|\chi_s^{(s)} - \xi^{(s)}\|_{L_1[-a,a]} \ge A, \tag{2.1}$$

where $0 < A \le a$, then

$$\|\chi_s - \xi\|_{L_1[-a,a]} \ge 2^{-s^2-4s-3}a^{s-1}A^2. \tag{2.2}$$

Proof: Denote

$$\delta_s(t) := \chi_s(t) - \xi(t), \quad t \in [-a,a].$$

It is clear from our assumptions that $\delta_s^{(s-1)}$ is decreasing and concave in $[-a,0]$, and it is increasing and concave in $[0,a]$. Assume first that

$$\max\{\delta_s^{(s-1)}(-a),\delta_s^{(s-1)}(a)\}\le 2^{-4}a^{-1}A^2. \tag{2.3}$$

Then by (2.1),

$$\begin{aligned}A&\le \|\delta_s^{(s)}\|_{L_1[-a,a]}\\ &=-\int_{-a}^{0}\delta_s^{(s)}(t)dt+\int_0^a\delta_s^{(s)}(t)dt\\ &=\delta_s^{(s-1)}(-a)-2\delta_s^{(s-1)}(0)+\delta_s^{(s-1)}(a),\end{aligned}$$

which by virtue of (2.3) implies

$$\delta_s^{(s-1)}(0)\le -A/2+2^{-4}a^{-1}A^2. \tag{2.4}$$

If

$$\zeta_{s-1}(t):=\begin{cases}2^{-4}a^{-1}A^2, & t\in[-a,-A/2],\\ |t|-A/2+2^{-4}a^{-1}A^2, & t\in[-A/2,A/2],\\ 2^{-4}a^{-1}A^2, & t\in[A/2,a],\end{cases}$$

then it follows by (2.4) that $\delta_s^{(s-1)}(0)\le\zeta_{s-1}(0)$. Since $|\delta_s^{(s)}(t)|\le 1$, and $|\zeta'_{s-1}(t)|=1$, $t\in[-A/2,A/2]$, $t\ne 0$, we conclude that the graph of $\delta_s^{(s-1)}$ in that interval is below that of ζ_{s-1}, i.e.,

$$\delta_s^{(s-1)}(t)\le\zeta_{s-1}(t),\quad t\in[-A/2,A/2]. \tag{2.5}$$

Also since $\delta_s^{(s-1)}$ is nonincreasing in $[-a,-A/2]$ and nondecreasing in $[A/2,a]$, it follows from (2.3) that

$$\delta_s^{(s-1)}(t)\le\zeta_{s-1}(t),\quad t\in[-a,-A/2]\cup[A/2,a],$$

which combined with (2.5) yields

$$\delta_s^{(s-1)}(t)\le\zeta_{s-1}(t),\quad t\in[-a,a]. \tag{2.6}$$

Let

$$\zeta_{s-2}(t):=\int_0^t\zeta_{s-1}(\tau)d\tau,\quad t\in[-a,a].$$

Then it is an odd function, it is nondecreasing in $[-a,-A/2+2^{-4}a^{-1}A^2]$, and it is nonicreasing in $[-A/2+2^{-4}a^{-1}A^2,0]$. At $t_{\text{max}}:=-A/2+2^{-4}a^{-1}A^2$ it has a local maximum and symmetrically, at $t_{\text{min}}:=A/2-2^{-4}a^{-1}A^2$ it has a local minimum. It is easy to calculate that

$$\zeta_{s-2}(-a)=(2^{-3}-2^{-4})A^2,\quad \zeta_{s-2}(a)=-(2^{-3}-2^{-4})A^2.$$

Hence,

$$\begin{aligned} \zeta_{s-2}(t) &\geq \zeta_{s-2}(-a) = 2^{-4}A^2, \quad t \in [-a/2, t_{\max}], \\ \zeta_{s-2}(t) &\leq \zeta_{s-2}(a) = -2^{-4}A^2, \quad t \in [t_{\min}, a/2]. \end{aligned} \tag{2.7}$$

Now

$$\int_0^t (\zeta_{s-1}(\tau) - \delta_s^{(s-1)}(\tau))d\tau = \zeta_{s-2}(t) - \delta_s^{(s-2)}(t) + \delta_s^{(s-2)}(0).$$

Thus by (2.6) and (2.7),

$$\begin{aligned} \delta_s^{(s-2)}(t) &\geq \delta_s^{(s-2)}(0) + \zeta_{s-2}(t) \\ &\geq \delta_s^{(s-2)}(0) + 2^{-4}A^2, \quad t \in [-a, t_{\max}], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \delta_s^{(s-2)}(t) &\leq \delta_s^{(s-2)}(0) + \zeta_{s-2}(t) \\ &\leq \delta_s^{(s-2)}(0) - 2^{-4}A^2, \quad t \in [t_{\min}, a]. \end{aligned} \tag{2.9}$$

If $\delta_s^{(s-2)}(0) \geq 0$, then by (2.8) we obtain

$$\delta_s^{(s-2)}(t) \geq 2^{-4}A^2, \quad t \in [-a, t_{\max}].$$

Otherwise, by (2.9) we get

$$\delta_s^{(s-2)}(t) \leq -2^{-4}A^2, \quad t \in [t_{\min}, a].$$

Hence we conclude that

$$\min_{t \in J_0} |\delta_s^{(s-2)}(t)| \geq 2^{-4}A^2, \tag{2.10}$$

where J_0 is either $[-a, t_{\max}]$ or $[t_{\min}, a]$. Our assumption that $A \leq a$ implies that $t_{\max} = -A/2 + 2^{-4}a^{-1}A^2 > -a/2$ and $t_{\min} = A/2 - 2^{-4}a^{-1}A^2 < a/2$, so that

$$|J_0| > a/2. \tag{2.11}$$

For $s = 2$, (2.10) and (2.11) yield (2.2). If $s > 2$, then we apply Lemma 1 and obtain by (2.10) and (2.11) that there exists an interval $J_{s-2} \subseteq J_0$ of length $|J_{s-2}| = 2^{-2s+4}|J_0| \geq 2^{-2s+3}a$ such that

$$\min_{t \in J_{s-2}} |\delta_s(t)| \geq 2^{-s^2+2s-4}a^{s-2}A^2.$$

This in turn implies

$$\|\chi_s - \xi\|_{L_1[-a,a]} \geq \|\delta_s\|_{L_1(J_{s-2})} \geq 2^{-s^2+1}a^{s-1}A^2,$$

and (2.2) has been proved for $s > 2$. On the other hand, if (2.3) fails, then due to symmetry, we may assume without loss of generality that

$$\chi_s^{(s-1)}(-a) - \xi^{(s-1)}(-a) > 2^{-4}a^{-1}A^2.$$

Suppose that

$$\chi_s^{(s-1)}(-a/2) - \xi^{(s-1)}(-a/2) > -2^{-4}a^{-1}A^2.$$

Then by the concavity of $\delta_s^{(s-1)}$ in $[-a,-a/2]$, we have

$$\delta_s^{(s-1)}(t) > -2^{-2}a^{-2}A^2(t+3a/4), \quad t \in [-a,-a/2].$$

In particular

$$\delta_s^{(s-1)}(t) > 2^{-5}a^{-1}A^2, \quad t \in [-a,-7a/8].$$

We apply Lemma 1 with $J_0 := [-a,-7a/8]$ and obtain an interval $J_{s-1} \subseteq J_0$ of length $|J_{s-1}| \geq 2^{-2s+2}|J_0| = 2^{-2s-1}a$, such that

$$\min_{t\in J_{s-1}} |\delta_s(t)| \geq 2^{-s^2-2s-2}a^{s-2}A^2.$$

Hence,

$$\|\chi_s - \xi\|_{L_1[-a,a]} \geq \|\delta_s\|_{L_1(J_{s-1})} \geq 2^{-s^2-4s-3}a^{s-1}A^2. \tag{2.12}$$

Otherwise

$$\chi_s^{(s-1)}(-a/2) - \xi^{(s-1)}(-a/2) \leq -2^{-5}a^{-1}A^2,$$

and since $\delta_s^{(s-1)}$ is nonincreasing in $[-a/2,0]$, we obtain

$$\delta_s^{(s-1)}(t) \leq -2^{-4}a^{-1}A^2, \quad t \in [-a/2,0].$$

Once more we apply Lemma 1, this time with $J_0 := [-a/2,0]$, and obtain an interval $J_{s-1} \subseteq J_0$ of length $|J_{s-1}| \geq 2^{-2s+2}|J_0| = 2^{-2s+1}a$, such that

$$\min_{t\in J_{s-1}} |\delta_s(t)| \geq 2^{-s^2-3}a^{s-2}A^2.$$

Hence,

$$\|\chi_s - \xi\|_{L_1[-a,a]} \geq \|\delta_s\|_{L_1(J_{s-1})} \geq 2^{-s^2-2s-2}a^{s-1}A^2. \tag{2.13}$$

Combining (2.12) and (2.13), we have proved (2.2) for $s > 2$ in this case too. This completes the proof of Lemma 2. ∎

An immediate consequence is

COROLLARY 1: *For $\tau \in \mathbb{R}$ and $b > 0$, denote*

$$\chi_{s,\tau,b}(t) := \frac{b}{s!}(t-\tau)_+^s, \quad t \in \mathbb{R}, \quad s \in \mathbb{N}.$$

Let $s > 1$ and $\psi \in C^s[\tau - a, \tau + a]$, $a > 0$, and assume that $\psi^{(s)}$ is nondecreasing and $0 \le \psi^{(s)}(t) \le b$, in $[\tau - a, \tau + a]$. Then, if

$$\|\chi_{s,\tau,b}^{(s)} - \psi^{(s)}\|_{L_1[\tau-a,\tau+a]} \ge A,$$

where $0 < A \le ab$, then

$$\|\chi_{s,\tau,b} - \psi\|_{L_1[\tau-a,\tau+a]} \ge 2^{-s^2-4s-3}a^{s-1}b^{-1}A^2.$$

In the sequel we use the standard notation for the unit vectors along the axes, namely,

$E^n := \{e^{(i)}\}_{i=1}^n$, $e^{(i)} := (0, \dots, 1, \dots, 0)$ with the 1 is standing in the ith entry,

and also we use

$\tilde{E}^n := \{\tilde{e}^{(i)}\}_{i=1}^n$, $\tilde{e}^{(1)} := (1, 1, \dots, 1), \tilde{e}^{(2)} := (0, 1, \dots, 1), \dots, \tilde{e}^{(n)} := (0, \dots, 0, 1)$.

Finally, the following was proved in [5, Lemma 4]

LEMMA D: *Let $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ be such that $m + 1 < n$, and let $1 \le p \le q \le \infty$. Denote by*

$$S_p^+(\tilde{E}^n) := \{v \mid v := \sum_{i=1}^n a_i \tilde{e}^{(i)},\ a = (a_1, \dots, a_n) \in \mathbb{R}^n,\ a_i \ge 0,\ i = 1, \dots, n, \\ \|a\|_{l_p^n} \le 1\}$$

the positive p-sector over the system $\tilde{E}^n$, and by

$$\Delta_+^1 := \{x = (x_1, \dots, x_n) \mid x_1 \le \dots \le x_n\}$$

the cone of vectors x with nondecreasing coordinates in $\mathbb{R}^n$. Then

$$d_m(S_p^+(\tilde{E}^n), \Delta_+^1)_{l_q^n} \ge \frac{1}{8}.$$

3. The upper bounds

Proof of the upper bounds: First take $s = r \geq 3$. It follows by Lemma C that there exists a quadratic spline $\sigma_{2,n}(x^{(r-3)};\cdot)$ with knots $t_i = t_{ni} := i/n$, $i = 0, \pm 1 \ldots, \pm n$, such that

$$x^{(r-1)}(t_{i-1}) \leq \sigma''_{2,n}(x^{(r-3)};t) \leq x^{(r-1)}(t_i), \quad t \in (t_{i-1}, t_i), \quad i = -n+1, \ldots, n,$$

and

$$\begin{aligned} &\|x^{(r-3)}(\cdot) - \sigma_{2,n}(x^{(r-3)};\cdot)\|_{L_\infty} \leq \frac{3}{2} n^{-3+1/p}, \\ (3.1) \qquad &\|x^{(r-2)}(\cdot) - \sigma'_{2,n}(x^{(r-3)};\cdot)\|_{L_\infty} \leq \frac{7}{2} n^{-2+1/p}, \\ &\|x^{(r-1)}(\cdot) - \sigma''_{2,n}(x^{(r-3)};\cdot)\|_{L_\infty} \leq n^{-1+1/p}. \end{aligned}$$

For $r = 3$, this spline provides us with the required upper bound. If $r > 3$, then by Taylor's formula, the spline

$$\sigma_{r-1,n}(x;t) := \sum_{l=0}^{r-4} \frac{1}{l!} x^{(l)}(0) t^l + \frac{1}{(r-4)!} \int_0^t \sigma_{2,n}(x^{(r-3)};\tau)(t-\tau)^{r-4} d\tau, \quad t \in I$$

yields

$$x(t) - \sigma_{r-1,n}(x;t) = \frac{1}{(r-4)!} \int_0^t (x^{(r-3)}(\tau) - \sigma_{2,n}(x^{(r-3)};\tau))(t-\tau)^{r-4} d\tau, \quad t \in I,$$

whence by (3.1) we obtain

$$\|x(\cdot) - \sigma_{r-1,n}(x;\cdot)\|_{L_\infty} \leq \frac{3}{(r-3)!} n^{-3+1/p}.$$

Thus the case $s = r \geq 3$ is proved.

Assume that $3 \leq s < r$. First we construct a spline $\sigma_{s,r,n}(x^{(s-3)};\cdot)$, which is not necessarily in $\Delta^s_+ L_q$, so that

$$\|x^{(s-3)}(\cdot) - \sigma_{s,r,n}(x^{(s-3)};\cdot)\|_{L_\infty} \leq c n^{r-s+1/p-3},$$

where $c = c(s,r,p)$. Then we will modify it a little so as to keep it close to x while at the same time be in $\Delta^s_+ L_q$. Denote by

$$I_i = I_{ni} := \begin{cases} [t_{i-1}, t_i], & i = 1, \ldots, n \\ [t_i, t_{i+1}], & i = -n, \ldots, -1 \end{cases}$$

the intervals of the partition. On each interval I_i, we define two complementary cubic splines φ_{*i} and φ_i^* as follows. For $i = 1, \ldots, n$, set

$$\varphi_{*i}^{(3)}(t) := \begin{cases} -32n^3, & t_{i-1} < t < t_{i-1} + 1/4n \\ 32n^3, & t_{i-1} + 1/4n < t < t_i - 1/4n \\ -32n^3 & t_i - 1/4n < t < t_i, \end{cases}$$

and

$$\varphi_i^{*(3)} := -\varphi_{*i}^{(3)}.$$

Now let

$$\begin{aligned} \varphi_{*i}''(t) &:= \int_{t_{i-1}}^t \varphi_{*i}^{(3)}(\tau)\,d\tau, \quad \varphi_i^{*\prime\prime}(t) := \int_{t_i}^t \varphi_i^{*(3)}(\tau)\,d\tau, \quad t \in I_i, \\ \varphi_{*i}'(t) &:= \int_{t_{i-1}}^t \varphi_{*i}''(\tau)\,d\tau, \quad \varphi_i^{*\prime}(t) := \int_{t_i}^t \varphi_i^{*\prime\prime}(\tau)\,d\tau, \quad t \in I_i, \\ \varphi_{*i}(t) &:= \int_{t_i}^t \varphi_{*i}'(\tau)\,d\tau, \quad \varphi_i^{*}(t) := \int_{t_{i-1}}^t \varphi_i^{*\prime}(\tau)\,d\tau, \quad t \in I_i. \end{aligned}$$

For $i = -n, \ldots, -1$ we set

$$\varphi_{*i}(t) := \varphi_{*,-i}(-t), \quad t \in I_i$$

and

$$\varphi_i^*(t) := \varphi_{-i}^*(-t), \quad t \in I_i.$$

All the above functions are piecewise cubic polynomials on the respective intervals, and it is readily seen that

$$\begin{aligned} &\varphi_{*i}(t_{i-1}) = \varphi_i^*(t_i) = 1, \quad \varphi_{*i}(t_i) = \varphi_i^*(t_{i-1}) = 0, \quad i = 1, \ldots, n, \\ &\varphi_{*i}(t_{i+1}) = \varphi_i^*(t_i) = 1, \quad \varphi_{*i}(t_i) = \varphi_i^*(t_{i+1}) = 0, \quad i = -n, \ldots, -1, \\ &\varphi_{*i}'(t_{i-1}) = \varphi_i^{*\prime}(t_i) = 0, \quad \varphi_{*i}'(t_i) = \varphi_i^{*\prime}(t_{i-1}) = 0, \quad i = 1, \ldots, n, \\ &\varphi_{*i}'(t_{i+1}) = \varphi_i^{*\prime}(t_i) = 0, \quad \varphi_{*i}'(t_i) = \varphi_i^{*\prime}(t_{i+1}) = 0, \quad i = -n, \ldots, -1, \\ &\varphi_{*i}''(t_{i-1}) = \varphi_i^{*\prime\prime}(t_i) = 0, \quad \varphi_{*i}''(t_i) = \varphi_i^{*\prime\prime}(t_{i-1}) = 0, \quad i = 1, \ldots, n, \\ &\varphi_{*i}''(t_{i+1}) = \varphi_i^{*\prime\prime}(t_i) = 0, \quad \varphi_{*i}''(t_i) = \varphi_i^{*\prime\prime}(t_{i+1}) = 0, \quad i = -n, \ldots, -1. \end{aligned} \tag{3.2}$$

Furthermore, for all $i = \pm 1, \ldots, \pm n$,

$$0 \le \varphi_{*i}(t) \le 1, \quad 0 \le \varphi_i^*(t) \le 1, \quad \text{and} \quad \varphi_{*i}(t) + \varphi_i^*(t) \equiv 1, \quad t \in I_i. \tag{3.3}$$

Thus, in particular,

$$\|\varphi_{*i}\|_{L_\infty(I_i)} = \|\varphi_i^*\|_{L_\infty(I_i)} = 1, \quad i = \pm 1, \ldots, \pm n. \tag{3.4}$$

Also

$$\begin{aligned}
&\|\varphi'_{*i}\|_{L_\infty(I_i)} = \|\varphi_i^{*\prime}\|_{L_\infty(I_i)} = 2n,\\
(3.5)\qquad &\|\varphi''_{*i}\|_{L_\infty(I_i)} = \|\varphi_i^{*\prime\prime}\|_{L_\infty(I_i)} = 8n^2,\\
&\|\varphi_{*i}^{(3)}\|_{L_\infty(I_i)} = \|\varphi_i^{*(3)}\|_{L_\infty(I_i)} = 32n^3, \quad i=\pm1,\dots,\pm n.
\end{aligned}$$

Let $0 \le k < r$ and assume $y \in C^k(I)$. For $1 \le i \le n$, let $\pi_{*,k}(y;i;t)$ and $\pi_k^*(y;i;t)$ denote the Taylor polynomials of degree k of y, expanded respectively, about the left-hand and right-hand endpoints of the interval I_i, that is,

$$\pi_{*,k}(y;i;t) := \sum_{l=0}^{k} \frac{1}{l!} y^{(l)}(t_{n,i-1})(t-t_{n,i-1})^l, \quad i=1,\dots,n,$$

$$\pi_k^*(y;i;t) := \sum_{l=0}^{k} \frac{1}{l!} y^{(l)}(t_i)(t-t_i)^l, \quad i=1,\dots,n.$$

Symmetrically, for $-n \le i \le -1$, let $\pi_{*,k}(y;i;t)$ and $\pi_k^*(y;i;t)$, $i=-n,\dots,-1$ denote the Taylor polynomials of degree k of y, expanded respectively, about the right-hand and left-hand endpoints of the interval I_i.

We are ready to define the approximating splines for $x \in \Delta_+^s W_p^r$, namely, for $i=\pm1,\dots,\pm n$, set

$$\sigma_{s,r,n}(x^{(s-3)};t) := \pi_{*,r-s+2}(x^{(s-3)};i;t)\varphi_{*i}(t) + \pi^*_{r-s+2}(x^{(s-3)};i;t)\varphi_i^*(t), \quad t \in I_i.$$

Evidently, $\sigma_{s,r,n}(x^{(s-3)};\cdot) \in C^2(I)$, and it is a polynomial of degree $\le r-s+5$ on each interval of the refined partition. We are going to prove that

$$(3.6)\qquad \|x^{(s-3)}(\cdot) - \sigma_{s,r,n}(x^{(s-3)};\cdot)\|_{L_\infty(I_i)} \le \frac{1}{(r-s+2)!} n^{-r+s+1/p-3}.$$

Indeed, it follows by (3.3) that for each $1 \le i \le n$,

$$\begin{aligned}
&\|x^{(s-3)}(\cdot) - \sigma_{r,n}(x^{(s-3)};\cdot)\|_{L_\infty(I_{ni})}\\
&= \|x^{(s-3)}(\cdot) - (\pi_{*,r-s+2}(x^{(s-3)};i;\cdot)\varphi_{*i}(\cdot) + \pi^*_{r-s+2}(x^{(s-3)};i;\cdot)\varphi_i^*(\cdot))\|_{L_\infty(I_i)}\\
&= \|(x^{(s-3)}(\cdot) - \pi_{*,r-s+2}(x^{(s-2)};i;\cdot))\varphi_{*i}(\cdot)\\
&\qquad\qquad + (x^{(s-3)}(\cdot) - \pi^*_{r-s+2}(x;i;\cdot))\varphi_i^*(\cdot)\|_{L_\infty(I_i)}\\
&\le \max\Big\{\|x^{(s-3)} - \pi_{*,r-s+2}(x^{(s-3)};i)\|_{L_\infty(I_i)}, \|x^{(s-3)} - \pi^*_{r-s+2}(x^{(s-3)};i)\|_{L_\infty(I_i)}\Big\}.
\end{aligned}$$

Now, by Taylor's formula and Hölder's inequality, we obtain for $t \in I_i$, $1 \le i \le n$,

$$\begin{aligned}|x^{(s-3)}(t) - \pi_{*,r-s+2}(x^{(s-3)};i;t)| &\le \frac{1}{(r-s+2)!}\int_{t_{i-1}}^{t} |x^{(r)}(\tau)|(t-\tau)^{r-s+2}d\tau \\ &\le \frac{1}{(r-s+2)!}\|x^{(r)}(\cdot)\|_{L_p(I_i)}|I_i|^{r-s-1/p+3} \\ &\le \frac{1}{(r-s+2)!}n^{-r+s+1/p-3}.\end{aligned}$$

Similarly

$$\begin{aligned}|x^{(s-3)}(t) - \pi^*_{r-s+2}(x^{(s-3)};i;t)| &\le \frac{1}{(r-s+2)!}\int_{t}^{t_i} |x^{(r)}(\tau)|(\tau-t)^{r-s+2}d\tau \\ &\le \frac{1}{(r-s+2)!}n^{-r+s+1/p-3}.\end{aligned}$$

Therefore

$$\|x^{(s-3)}(\cdot) - \pi_{*,r-s+2}(x^{(s-3)};i;\cdot)\|_{L_\infty(I_i)} \le \frac{1}{(r-s+2)!}n^{-r+s+1/p-3}, \tag{3.7}$$

and

$$\|x^{(s-3)}(\cdot) - \pi^*_{r-s+2}(x^{(s-3)};i;\cdot)\|_{L_\infty(I_i)} \le \frac{1}{(r-s+2)!}n^{-r+s+1/p-3}, \tag{3.8}$$

and (3.6) is established for $i = 1, \dots, n$. For $i = -1, \dots, -n$ the proof is similar.

In the same way we have for $i = \pm 1, \dots, \pm n$,

$$\begin{aligned}&\|x^{(s-3+m)}(\cdot) - \pi^{(m)}_{*,r-s+2}(x^{(s-3)};i;\cdot)\|_{L_\infty(I_i)} \\ &= \|x^{(s-3+m)}(\cdot) - \pi_{*,r-s+2-m}(x^{(s-3+m)};i;\cdot)\|_{L_\infty(I_i)} \\ &\le \frac{1}{(r-s+2-m)!}n^{-r+s+1/p-3+m}, \quad m = 1,2,3\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\|x^{(s-3+m)}(\cdot) - \pi^*_{r-s+2}{}^{(m)}(x^{(s-3)};i;\cdot)\|_{L_\infty(I_i)} \\ &= \|x^{(s-3+m)}(\cdot) - \pi^*_{r-s+2-m}(x^{(s-3+m)};i;\cdot)\|_{L_\infty(I_i)} \\ &\le \frac{1}{(r-s+2-m)!}n^{-r+s+1/p-3+m}, \quad m = 1,2,3.\end{aligned} \tag{3.10}$$

Now, for the third derivative of $\sigma_{s,r,n}$, which exists a.e. in I_i, $1 \le |i| \le n$, we obtain by (3.3) through (3.5),

$$\begin{aligned}
&\sigma^{(3)}_{s,r,n}(x^{(s-3)};t)\\
&=\sum_{m=0}^{3}\binom{3}{m}(\pi^{(m)}_{*,r-s+2}(x^{(s-3)};i;t)\varphi^{(3-m)}_{*i}(t)+\pi^{*}_{r-s+2}{}^{(m)}(x^{(s-3)};i;t)\varphi^{*(3-m)}_{i}(t))\\
&=x^{(s)}(t)-\sum_{m=0}^{3}\binom{3}{m}(x^{(s-3+m)}(t)-\pi^{(m)}_{*,r-s+2}(x^{(s-3)};i;t))\varphi^{(3-m)}_{*i}(t)\\
&\quad-\sum_{m=0}^{3}\binom{3}{m}(x^{(s-3+m)}(t)-\pi^{*}_{r-s+2}{}^{(m)}(x^{(s-3)};i;t))\varphi^{*(3-m)}_{i}(t)\\
&=x^{(s)}(t)-\sum_{m=0}^{3}\binom{3}{m}(x^{(s-3+m)}(t)-\pi_{*,r-s+2-m}(x^{(s-3+m)};i;t))\varphi^{(3-m)}_{*i}(t)\\
&\quad-\sum_{m=0}^{3}\binom{3}{m}(x^{(s-3+m)}(t)-\pi^{*}_{r-s+2-m}(x^{(s-3+m)};i;t))\varphi^{*(3-m)}_{i}(t).
\end{aligned}$$

This, together with (3.7) through (3.10), yields

(3.11)
$$\|x^{(s)}(\cdot)-\sigma^{(3)}_{s,r,n}(x^{(s-3)};\cdot)\|_{L_\infty(I_i)}\le \hat{c}n^{-r+s+1/p},\quad \text{a.e. } t\in I_i,\quad 1\le|i|\le n,$$

where
$$\hat{c}:=\frac{64}{(r-s+2)!}+\frac{48}{(r-s+1)!}+\frac{12}{(r-s)!}+\frac{1}{(r-s-1)!}.$$

We have to modify the spline $\sigma_{s,r,n}(x^{(s-3)};\cdot)$ so that its second derivative is monotone. To this end, we take

$$\hat{m}=\hat{m}(s,r):=928(r-s), \tag{3.12}$$

and set

$$t_{i,k}=t_{n,i,k}:=\begin{cases} t_{i-1}+k(\hat{m}n)^{-1}, & k=0,1,\dots,\hat{m},\quad i=1,\dots,n,\\ t_{i+1}-k(\hat{m}n)^{-1}, & k=0,1,\dots,\hat{m},\quad i=-1,\dots,-n.\end{cases}$$

Let

$$I_{i,k}=I_{n,i,k}:=\begin{cases}[t_{i,k-1},t_{i,k}], & k=1,\dots,\hat{m},\quad i=1,\dots,n,\\ [t_{i,k},t_{i,k-1}], & k=1,\dots,\hat{m},\quad i=-1,\dots,-n.\end{cases}$$

The sth derivative $x^{(s)}$ is called **small** in I_i, $1\le|i|\le n$ if there exist at least $2(r-s)(\le\hat{m})$ subintervals I_{i,k_j}, and points $\tau_{i,k_j}\in I_{i,k_j}$, such that

$$x^{(s)}(\tau_{i,k_j})\le 2\hat{c}n^{-r+s+1/p}. \tag{3.13}$$

Otherwise $x^{(s)}$ is called **big** in I_i.

If $x^{(s)}(\cdot)$ is small in I_{i_*}, let $J_* := [t_{\nu_*}, t_{\nu^*}]$, $-n \le \nu_* < \nu^* \le n$, be the biggest interval containing I_{i_*}, so that $x^{(s)}$ is small in I_ν, $\nu_* \le \nu \le \nu^* - 1$. Since in I_ν there are at least $r-s$ disjoint points τ_{ν,k_j}, any two of which are at distance of at least $(\hat{m}n)^{-1}$. Applying Lemma A and (3.13), we obtain by Hölder's inequality,

$$\begin{aligned}
&\|x^{(s)}\|_{L_\infty(I_\nu)}\\
&\le \frac{r-s}{(r-s-1)!}\hat{m}^{\frac{(r-s)(r-s-1)}{2}}\\
&\qquad\times \Big(\max|x^{(s)}(\tau_{\nu,k_j})| + \frac{1}{(r-s-1)!}n^{-r+s-1}\|x^{(r)}\|_{L_1(I_\nu)}\Big)\\
&\le \frac{r-s}{(r-s-1)!}\hat{m}^{\frac{(r-s)(r-s-1)}{2}}\\
&\qquad\times \Big(2\hat{c}n^{-r+s+1/p} + \frac{1}{(r-s-1)!}n^{-r+s+1/p}\|x^{(r)}\|_{L_p(I_\nu)}\Big)\\
&\le \frac{r-s}{(r-s-1)!}\hat{m}^{\frac{(r-s)(r-s-1)}{2}}\Big(2\hat{c} + \frac{1}{(r-s-1)!}\Big)n^{-r+s+1/p}.
\end{aligned}$$

Hence

$$\|x^{(s)}\|_{L_\infty(J_*)} \le c_* n^{-r+s+1/p}, \tag{3.14}$$

where

$$c_* := \frac{r-s}{(r-s-1)!}\hat{m}^{\frac{(r-s)(r-s-1)}{2}} 3\hat{c}.$$

We divide J_* into subintervals $J_{*j} := [\tau_{*,j-1}, \tau_{*,j}]$, $1 \le j \le J = J(n, m_*, J_*)$, of length $|J_{*j}| = (m_* n)^{-1}$, where

$$m_* := 2784(r-s)\hat{m}^{\frac{(r-s)(r-s-1)}{2}}. \tag{3.15}$$

By virtue of Lemma 2 there exists a quadratic spline $\sigma_{2,n}(x^{(s-3)};\cdot;J_*)$ with knots τ_{*j} such that

$$x^{(s-1)}(\tau_{*,j-1}) \le \sigma''_{2,n}(x^{(s-3)};t;J_*) \le x^{(s-1)}(\tau_{*j}), \quad t \in (\tau_{*,j-1}, \tau_{*j}),$$

for all $j = 1, \ldots, J(n, m_*, J_*)$, and

$$\begin{aligned}
&\|x^{(s-3)}(\cdot) - \sigma_{2,m_*n}(x^{(s-3)};\cdot;J_*)\|_{L_\infty(J_*)} \le \frac{3}{2}(m_*n)^{-3}\|x^{(s)}\|_{L_\infty(J_*)},\\
&\|x^{(s-2)}(\cdot) - \sigma'_{2,m_*n}(x^{(s-3)};\cdot;J_*)\|_{L_\infty(J_*)} \le \frac{7}{2}(m_*n)^{-2}\|x^{(s)}\|_{L_\infty(J_*)},\\
&\|x^{(s-1)}(\cdot) - \sigma''_{2,m_*n}(x^{(s-3)};\cdot;J_*)\|_{L_\infty(J_*)} \le (m_*n)^{-1}\|x^{(s)}\|_{L_\infty(J_*)}.
\end{aligned}$$

This in turn yields, by (3.14),

$$
\begin{aligned}
&\|x^{(s-3)}(\cdot)-\sigma_{2,m_*n}(x^{(s-3)};\cdot;J_*)\|_{L_\infty(J_*)}\le m_*^{-3}\frac{3}{2}c_*n^{-r+s+1/p-3},\\
&\|x^{(s-2)}(\cdot)-\sigma'_{2,m_*n}(x^{(s-3)};\cdot;J_*)\|_{L_\infty(J_*)}\le m_*^{-2}\frac{7}{2}c_*n^{-r+s+1/p-2},\\
&\|x^{(s-1)}(\cdot)-\sigma''_{2,m_*n}(x^{(s-3)};\cdot;J_*)\|_{L_\infty(J_*)}\le m_*^{-1}c_*n^{-r+s+1/p-1}.
\end{aligned}
\tag{3.16}
$$

We replace $\sigma_{s,r,n}(x^{(s-3)};t)$ on J_* with $\sigma_{2,n}(x^{(s-3)};\cdot;J_*)$, and set

$$\bar\sigma_{s,r,n}(x^{(s-3)};t):=\sigma_{2,n}(x^{(s-3)};t;J_*),\quad t\in J_*.$$

There may be a few intervals of the type J_*, all of course being mutually disjoint. In the extreme case it may be that $J_*=I$, in which case of course we are done. Otherwise $x^{(s)}(\cdot)$ is big in some subintervals I_i, so let I_{i_0} be such an interval. Let $\{I_{i_0,k_j}\subset I_{i_0}\}$, be the collections of all $0\le m=m(I_{i_0})<2(r-s)$ subintervals each of which contains a point τ_{i_0,k_j} such that

$$x^{(s)}(\tau_{i_0,k_j})\le 2\hat c n^{-r+s+1/p}. \tag{3.17}$$

Define

$$\xi_{i_0}^{(3)}(x^{(s-3)};t):=\begin{cases}2\hat c n^{-r+s+1/p}, & t\in I_{i_0,k_j}\\ 0, & \text{otherwise}\end{cases}$$

(3.18) and

$$\xi_{i_0}(x^{(s-3)};t):=\frac{1}{2}\int_{\bar t_{i_0}}^{t}\xi_{i_0}^{(3)}(x^{(s-3)};\tau)(t-\tau)^2d\tau,$$

where $\bar t_{i_0}:=\frac{1}{2}(t_{i_0-1}+t_{i_0})$. It follows by (3.17) that

$$|\xi_i^{(l)}(t_i)|,|\xi_i^{(l)}(t_{i-1})|\le\hat m^{-1}2(r-s)2\hat c n^{-r+s+1/p-3+l},\quad l=0,1,2. \tag{3.19}$$

Also note that on all other subintervals of I_{i_0} we have

$$x^{(s)}(t)>2\hat c n^{-r+s+1/p}. \tag{3.20}$$

Now set

$$\bar\sigma_{s,r,n}(x^{(s-3)};t):=\sigma_{s,r,n}(x^{(s-3)};t)+\xi_{i_0}(x^{(s-3)};t),\quad t\in I_{i_0}.$$

This defines spline pieces with possible discontinuities at the points t_i and we need to join them smoothly together. To this end, write

$$\phi_{*0i}(t):=\varphi_{*i}(t),\quad \phi_{0i}^*(t):=\varphi_i^*(t),\quad t\in I_i,\quad i=\pm1,\dots,\pm n,$$

and let

$$\phi_{*1i}(t) := -\frac{1}{n}\varphi_{*i}(\frac{1}{2}(t-t_i)+t_i), \quad \phi_{1i}^*(t) := \frac{1}{n}\varphi_i^*(\frac{1}{2}(t-t_{i-1})+t_{i-1}),$$
$$\phi_{*2i}(t) := \frac{2}{n^2}\varphi_{*i}(\frac{1}{4}(t-t_i)+t_i), \quad \phi_{2i}^*(t) := \frac{2}{n^2}\varphi_i^*(\frac{1}{4}(t-t_{i-1})+t_{i-1}),$$
$$t \in I_i, \quad 1 \le i \le n$$

and

$$\phi_{*li}(t) := \phi_{*l,-i}(-t), \quad \phi_{li}^*(t) := \phi_{l,-i}^*(-t), \quad l=0,1,2, \quad -n \le i \le -1.$$

Let $1 \le i \le n$. Straightforward computations yield

$$\begin{array}{lll} \phi_{*1i}(t_{i-1}) = -(2n)^{-1}, & \phi_{*1i}(t_i) = 0, & \|\phi_{*1i}\|_{L_\infty(I_i)} = (2n)^{-1}, \\ \phi_{1i}^*(t_{i-1}) = 0, & \phi_{1i}^*(t_i) = (2n)^{-1}, & \|\phi_{1i}^*\|_{L_\infty(I_i)} = (2n)^{-1}, \\ \phi'_{*1i}(t_{i-1}) = 1, & \phi'_{*1i}(t_i) = 0, & \|\phi'_{*1i}\|_{L_\infty(I_i)} = 1, \\ {\phi_{1i}^*}'(t_{i-1}) = 0, & {\phi_{1i}^*}'(t_i) = 1, & \|{\phi_{1i}^*}'\|_{L_\infty(I_i)} = 1, \\ \phi''_{*1i}(t_{i-1}) = 0, & \phi''_{*1i}(t_i) = 0, & \|\phi''_{*1i}\|_{L_\infty(I_i)} = 2n, \\ {\phi_{1i}^*}''(t_{i-1}) = 0, & {\phi_{1i}^*}''(t_i) = 0, & \|{\phi_{1i}^*}''\|_{L_\infty(I_i)} = 2n, \end{array} \tag{3.21}$$

and

$$\begin{array}{lll} \phi_{*2i}(t_{i-1}) = \frac{1}{6}n^{-2}, & \phi_{*2i}(t_i) = 0, & \|\phi_{*2i}\|_{L_\infty(I_i)} = \frac{1}{6}n^{-2}, \\ \phi_{2i}^*(t_{i-1}) = 0, & \phi_{2i}^*(t_i) = \frac{1}{6}n^{-2}, & \|\phi_{2i}^*\|_{L_\infty(I_i)} = \frac{1}{6}n^{-2}, \\ \phi'_{*2i}(t_{i-1}) = -(2n)^{-1}, & \phi'_{*2i}(t_i) = 0, & \|\phi'_{*2i}\|_{L_\infty(I_i)} = (2n)^{-1}, \\ {\phi_{2i}^*}'(t_{i-1}) = 0, & {\phi_{2i}^*}'(t_i) = (2n)^{-1}, & \|{\phi_{2i}^*}'\|_{L_\infty(I_i)} = (2n)^{-1}, \\ \phi''_{*2i}(t_{i-1}) = 1, & \phi''_{*2i}(t_i) = 0, & \|\phi''_{*2i}\|_{L_\infty(I_i)} = 1, \\ {\phi_{2i}^*}''(t_{i-1}) = 0, & {\phi_{2i}^*}''(t_i) = 1, & \|{\phi_{2i}^*}''\|_{L_\infty(I_i)} = 1. \end{array} \tag{3.22}$$

Also,

$$\begin{array}{c} \|\phi_{*1i}^{(3)}\|_{L_\infty(I_i)} = 4n^2, \|\phi_{*2i}^{(3)}\|_{L_\infty(I_i)} = n, \\ \|{\phi_{1i}^*}^{(3)}\|_{L_\infty(I_i)} = 4n^2, \|{\phi_{2i}^*}^{(3)}\|_{L_\infty(I_i)} = n. \end{array} \tag{3.23}$$

Since $\bar{\sigma}_{s,r,n}(x^{(s-3)},\cdot)$ may have jumps at the points t_i, let

$$\delta_{ki} := \begin{cases} \lim_{t\to t_i+} \bar{\sigma}_{s,r,n}^{(k)}(x^{(s-3)};t) - \lim_{t\to t_i-} \bar{\sigma}_{s,r,n}^{(k)}(x^{(s-3)};t), \\ \qquad k=0,1,2, \quad -n+1 \le i \le n-1, \\ 0, \quad k=0,1,2, \quad i = \pm n, \end{cases}$$

and define a correcting cubic spline on the intervals where $x^{(s)}$ is big. (In view of the different indices ascribed to the endpoints of the intervals, we only describe how we deal with I_{i_0}, $i_0 \ge 0$, where $x^{(s)}$ is big. The other intervals are handled in a similar way.) Thus, suppose $x^{(s)}$ is big also in I_{i_0-1}; then set

$$\zeta_{i_0}(x^{(s-3)};t) := \sum_{l=0}^{2}(\lambda_{*li_0}\phi_{*li_0}(t) + \lambda_{li_0}^*\phi_{li_0}^*(t)) \tag{3.24}$$

such that

$$\zeta_{i_0}^{(k)}(x^{(s-3)};t_{i_0-1})=0 \quad\text{and}\quad \zeta_{i_0}^{(k)}(x^{(s-3)};t_{i_0})=\delta_{k,i_0},\quad k=0,1,2. \tag{3.25}$$

If, on the other hand, $x^{(s)}$ is big in I_{i_0} but small in I_{i_0-1}, then set ζ_{i_0} as in (3.24) such that

$$\zeta_{i_0}^{(k)}(x^{(s-3)};t_{i_0-1})=-\delta_{k,i_0-1} \quad\text{and}\quad \zeta_{i_0}^{(k)}(x^{(s-3)};t_{i_0})=\delta_{k,i_0},\quad k=0,1,2. \tag{3.26}$$

The existence of $\zeta_{i_0}(x^{(s-3)};\cdot)$ in both above cases is guaranteed by (3.2), (3.21) and (3.22). In fact, solving the system of linear equations (3.25) in the former case we obtain $\lambda_{*li_0}=0$, $l=0,1,2$, and solving equations (3.26) in the latter case we obtain

$$\begin{aligned}\lambda_{*0,i_0}&=-\delta_{0,i_0-1}-(2n)^{-1}\delta_{1,i_0-1}-\frac{1}{3}(2n)^{-2}\delta_{2,i_0-1},\\ \lambda_{*1,i_0}&=-\delta_{1,i_0-1}-(2n)^{-1}\delta_{2,i_0-1},\\ \lambda_{*2,i_0}&=-\delta_{2,i_0-1}.\end{aligned} \tag{3.27}$$

In both cases

$$\begin{aligned}\lambda^*_{0,i_0}&=\delta_{0,i_0}-(2n)^{-1}\delta_{1,i_0}+\frac{1}{3}(2n)^{-2}\delta_{2,i_0},\\ \lambda^*_{1,i_0}&=\delta_{1,i_0}-(2n)^{-1}\delta_{2,i_0},\\ \lambda^*_{2,i_0}&=\delta_{2,i_0}.\end{aligned} \tag{3.28}$$

Denote

$$\tilde\sigma_{s,r,n}(x^{(s-3)};t):=\bar\sigma_{s,r,n}(x^{(s-3)};t)+\zeta_{i_0}(x^{(s-3)};t),\quad t\in I_{i_0}.$$

Clearly $\tilde\sigma_{s,r,n}(x^{(s-3)};\cdot)\in C^1(I)$. Furthermore, $\tilde\sigma''_{s,r,n}(x^{(s-3)};\cdot)$ exists and is continuous except perhaps at the points τ_{*j} of the intervals J_*, and in particular it is continuous at all points t_i, $1\le i<n$. We will show that $\tilde\sigma''_{s,r,n}(x^{(s-3)};\cdot)$ is nondecreasing on I. Indeed, it suffices to prove that $\tilde\sigma^{(3)}_{s,r,n}(x^{(s-3)};\cdot)$, which exists a.e. in I_{i_0}, is nonnegative there. By our construction,

$$\begin{aligned}\tilde\sigma^{(3)}_{s,r,n}(x^{(s-3)};t)&=\sigma^{(3)}_{s,r,n}(x^{(s-3)};t)+\xi^{(3)}_{i_0}(x^{(s-3)};t)+\zeta^{(3)}_{i_0}(x^{(s-3)};t)\\ &=x^{(s)}(t)+\xi^{(3)}_{i_0}(x^{(s-3)};t)-(x^{(s)}(t)-\sigma^{(3)}_{s,r,n}(x^{(s-3)};t))\\ &\qquad+\zeta^{(3)}_{i_0}(x^{(s-3)};t),\\ &\quad t\in I_{i_0}.\end{aligned}$$

By (3.18) and (3.20) we are guaranteed that

$$x^{(s)}(t)+\xi^{(3)}_{i_0}(x^{(s-3)};t)\ge 2\hat c n^{-r+s+1/p},\quad \text{a.e. in } I_{i_0}.$$

Hence (3.11) yields

$$(3.29)\quad \tilde{\sigma}^{(3)}_{s,r,n}(x^{(s-3)};t) \geq 2\hat{c}n^{-r+s+1/p} - \hat{c}n^{-r+s+1/p} - \|\zeta^{(3)}_{i_0}(x^{(s-3)};\cdot)\|_{L_\infty(I_{i_0})}.$$

Thus it remains to estimate the third term. Now by (3.24), (3.27) and (3.28), in the worst case,

$$\begin{aligned}\|\zeta^{(3)}_{i_0}(x^{(s-3)};\cdot)\|_{L_\infty(I_{i_0})} \leq & (|\delta_{0,i_0-1}| + (2n)^{-1}|\delta_{1,i_0-1}| \\ & \qquad + \frac{1}{3}(2n)^{-2}|\delta_{2,i_0-1}|)\|\phi^{(3)}_{*0,i_0}(\cdot)\|_{L_\infty(I_{i_0})} \\ & + (|\delta_{0,i_0}| + (2n)^{-1}|\delta_{1,i_0}| \\ & \qquad + \frac{1}{3}(2n)^{-2}|\delta_{2,i_0}|)\|{\phi^*_{0,i_0}}^{(3)}(\cdot)\|_{L_\infty(I_{i_0})} \\ & + (|\delta_{1,i_0-1}| + (2n)^{-1}|\delta_{2,i_0-1}|)\|\phi^{(3)}_{*1i_0}(\cdot)\|_{L_\infty(I_{i_0})} \\ & + (|\delta_{1,i_0}| + (2n)^{-1}|\delta_{2,i_0}|)\|{\phi^*_{1,i_0}}^{(3)}(\cdot)\|_{L_\infty(I_{i_0})} \\ & + |\delta_{2,i_0-1}|\|\phi^{(3)}_{*2,i_0}(\cdot)\|_{L_\infty(I_{i_0})} \\ & + |\delta_{2,i_0}|\|{\phi^*_{2,i_0}}^{(3)}(\cdot)\|_{L_\infty(I_{i_0})}.\end{aligned}$$

By virtue of (3.5) and (3.23) we obtain

$$(3.30)\quad \begin{aligned}&\|\zeta^{(3)}_{i_0}(x^{(s-3)};\cdot)\|_{L_\infty(I_{i_0})} \\ &\leq 32n^3(|\delta_{0,i_0-1}| + |\delta_{0,i_0}|) + 20n^2(|\delta_{1,i_0-1}| + |\delta_{1,i_0}|) + \frac{17}{3}n(|\delta_{2,i_0-1}| + |\delta_{2,i_0}|).\end{aligned}$$

In order to estimate the jumps at t_i, we observe that the original spline $\sigma_{r,s,n}(x^{(s-3)};\cdot) \in C^2(I)$ thus contributes nothing to the jumps. Moreover,

$$(3.31)\quad \lim_{t\to t_i} \sigma^{(k)}_{r,s,n}(x^{(s-3)};t) = x^{(k)}(t_i), \quad k = 0,1,2.$$

Hence, if $x^{(s)}$ is big both in I_{i_0} and in I_{i_0+1}, then by (3.19),

$$(3.32)\quad |\delta_{l,i_0}| \leq |\xi^{(l)}_{i_0}(t_{i_0})| + |\xi^{(l)}_{i_0+1}(t_{i_0})| \leq \hat{m}^{-1}4(r-s)2\hat{c}n^{-r+s+1/p-3+l}, \quad l = 0,1,2.$$

If, on the other hand, $x^{(s)}$ is big in I_{i_0} and small either in I_{i_0+1}, or in I_{i_0-1} or in both, then by (3.31) we have

$$|\delta_{l,i_0}| \leq \|x^{(s-l)}(\cdot) - \sigma^{(l)}_{2,n}(x^{(s-3)};\cdot;J^1_*)\|_{L_\infty(J^1_*)} + |\xi^{(l)}_{i_0}(t_{i_0})|, \quad l = 0,1,2$$

or

$$|\delta_{l,i_0-1}| \leq \|x^{(s-l)}(\cdot) - \sigma^{(l)}_{2,n}(x^{(s-3)};\cdot;J^2_*)\|_{L_\infty(J^2_*)} + |\xi^{(l)}_{i_0}(t_{i_0-1})|, \quad l = 0,1,2,$$

respectively, or both, where $J_*^1 \supset I_{i_0+1}$ or $J_*^2 \supset I_{i_0-1}$, respectively. By (3.16) and again (3.19), we obtain

$$|\delta_{l,i_0}|, |\delta_{l,i_0-1}| \leq (m_*^{-3+l} 4c_* + \hat{m}^{-1} 2(r-s) 2\hat{c}) n^{-r+s+1/p-3+l},$$

which together with (3.32) yields that in all cases,

$$|\delta_{l,i_0}|, |\delta_{l,i_0-1}| \leq (m_*^{-3+l} 4c_* + \hat{m}^{-1} 8(r-s)\hat{c}) n^{-r+s+1/p-3+l}, \quad l = 0, 1, 2.$$

Our choice of m_* and $\hat{m}$ (see (3.12) and (3.15)) gives

$$|\delta_{l,i_0-1}|, |\delta_{l,i_0}| \leq \frac{\hat{c}}{116} n^{-r+s+1/p-3+l},$$

and combined with (3.29) and (3.30) proves that $\tilde{\sigma}_{s,r,n}^{(3)}(x^{(s-3)}; t) \geq 0$, $t \in I_{i_0}$, as we have asserted.

Finally, the same computations yield

$$\|x^{(s-3)}(\cdot) - \tilde{\sigma}_{s,r,n}(x^{(s-3)}; \cdot)\|_{L_\infty} \leq c n^{-r+s+1/p-3}, \tag{3.33}$$

where $c = c(s, r, p)$.

If $3 = s < r$, then we set

$$\sigma_{s,r,n}(x; t) := \tilde{\sigma}_{s,r,n}(x^{(s-3)}; t), \quad t \in I.$$

If $3 < s < r$, then we set

$$\sigma_{s,r,n}(x; t) := \sum_{k=0}^{s-4} \frac{1}{k!} x^{(k)}(0) t^k + \frac{1}{(s-3)!} \int_0^t \tilde{\sigma}_{s,r,n}(x^{(s-3)}; \tau)(t-\tau)^{s-3} d\tau, \quad t \in I.$$

Then

$$x(t) - \sigma_{s,r,n}(x; t) = \frac{1}{(s-3)!} \int_0^t (x^{(s-3)}(\tau) - \tilde{\sigma}_{s,r,n}(x^{(s-3)}; \tau))(t-\tau)^{s-3} d\tau, \quad t \in I$$

and, by (3.33),

$$\begin{aligned} \|x(\cdot) - \sigma_{s,r,n}(x; \cdot)\|_{L_\infty} &\leq \frac{1}{(s-3)!} \|x^{(s-3)}(\cdot) - \tilde{\sigma}_{s,r,n}(x^{(s-3)}; \cdot)\|_{L_\infty} \\ &\leq c n^{-r+s+1/p-3}, \end{aligned}$$

where $c = c(s, r, p)$. Evidently $\sigma_{s,r,n}^{(s-2)}(x; \cdot) \in C(I)$ and its derivative $\sigma_{s,r,n}^{(s-1)}(x; \cdot)$ is nondecreasing in I. Thus $\sigma_{s,r,n}(x; \cdot) \in \Delta_+^s L_q$, $1 \leq q \leq \infty$, and the upper bounds are proved for all $3 \leq s \leq r$.

If $s = r+1$, $r \geq 2$, then $x^{(r-1)} \in \Delta_+^2 W_1^1$. It was proved in [5] that there exists a convex piecewise linear function $\sigma(x^{(r-1)};\cdot)$ with $2n+1$ knots such that

$$\|x^{(r-1)}(\cdot) - \sigma_{1,n}(x^{(r-1)};\cdot)\|_{L_1} \leq cn^{-2},$$

where c is an absolute constant. Set

$$\sigma_{r,n}(x;t) := \sum_{k=0}^{r-2} \frac{1}{k!} x^{(k)}(0)t^k + \frac{1}{(r-2)!}\int_0^t \sigma_{1,n}(x^{(r-1)};\tau)(t-\tau)^{r-2}d\tau, \quad t \in I,$$

which evidently satisfies $\sigma_{r,n}(x;\cdot) \in \Delta_+^{r+1}L_q$. Then

$$x(t) - \sigma_{r,n}(x;t) = \frac{1}{(r-2)!}\int_0^t (x^{(r-1)}(\tau) - \sigma_{1,n}(x;\tau))(t-\tau)^{r-2}d\tau, \quad t \in I.$$

Hence

$$\|x(\cdot) - \sigma_{r,n}(x;\cdot)\|_{L_\infty} \leq \frac{1}{(r-2)!}\|x^{(r-1)}(\cdot) - \sigma_{1,n}(x^{(r-1)};\cdot)\|_{L_1} \leq cn^{-2},$$

and the proof of the upper bounds is complete. ∎

4. The lower bounds

Proof of the lower bounds: In order to prove the lower bound, we let

$$\phi_0(t) := \begin{cases} 1, & t \in [-1,1] \\ 0, & t \in \mathbb{R}\setminus[-1,1] \end{cases}$$

and define by induction

$$\phi_s(t) := \int_{t-1}^t \phi_{s-1}(2\tau+1)\,d\tau, \quad t \in \mathbb{R}, \quad s \in \mathbb{N}.$$

It follows that for all $s \in \mathbb{Z}_+$, ϕ_s is even, $\phi_s \geq 0$, $\phi_s(t) = 0, t \in \mathbb{R}\setminus[-1,1]$, $|\phi_s^{(s)}(t)| = 2^{s-1}$, in $[-1,1]$ except for a few dyadic points with denominator 2^{-s+1}, and

$$\phi_s(0) = \|\phi_s\|_{L_\infty} = \int_{-1}^1 \phi_s(t)dt = 2^{-s+1}, \quad s \in \mathbb{N}.$$

For $N \in \mathbb{N}$, write $\phi_{s,N}(t) := N^{-s}\phi_s(Nt)$, and for

$$\tau_{N,i} := -\frac{1}{4} + \frac{i}{2N}, \quad i = 0,1,\ldots,N,$$
$$\bar{\tau}_{N,i} := -\frac{1}{4} + \frac{2i-1}{4N}, \quad i = 1,\ldots,N,$$

let

$$\phi_{s,N,i}(t) := \phi_{s,N}(t-\bar\tau_{N,i}), \quad i=1,\dots,N, \quad s\in\mathbb{Z}_+.$$

Finally, for $1\le p\le\infty$, set

$$\phi_{p,s,N,i}(t) := 2^{-s+1-1/p}N^{1/p}\phi_{s,N,i}(t), \quad s\in\mathbb{Z}_+.$$

Clearly, $\phi_{p,s,N,i}(t)$ is symmetric about $\bar\tau_{N,i}$, $\phi_{p,s,N,i}(t)=0$, $t\notin[\tau_{N,i-1},\tau_{N,i}]$, and

$$\phi_{p,0,N,i}(\bar\tau_{N,i}) = 2^{1-1/p}N^{1/p}, \quad \phi_{p,s,N,i}(\bar\tau_{N,i}) = 2^{-2s+2-1/p}N^{-s+1/p}. \tag{4.1}$$

Also

$$\|\phi^{(s)}_{p,s,N,i}\|_{L_p} = 1, \quad s\in\mathbb{Z}_+. \tag{4.2}$$

We are ready to construct the system of vectors that will yield the lower bound. We fix some $k\in\mathbb{N}$, $k>2$ to be prescribed, and set

$$\begin{gathered}\psi_{p,r,s,k,N,i}(t) := \int_{-1}^{t}\int_{-1}^{t_1}\cdots\int_{-1}^{t_{s-1}}\phi_{p,r-s,kN,k(i-1)+1}(t_s)dt_s\cdots dt_1,\\ i=1,\dots,N, \quad t\in[-1,1].\end{gathered} \tag{4.3}$$

Then it is s-convex in $I=[-1,1]$ and, by (4.2), belongs to $\Delta^s_+W^r_p$. Denote the system $\Psi^N_{p,r,s,k} := \{\psi_{p,r,s,k,N,i}(\cdot)\}_{i=1}^N$, and let

$$S^+_p(\Psi^N_{p,r,s,k}) := \{x=\sum_{i=1}^N a_i\psi_{p,r,s,k,N,i} \mid a_i\ge 0, \sum_{i=1}^N a_i^p\le 1\}$$

be the positive p-sector over this system. Then $S^+_p(\Psi^N_{p,r,s,k})\subset\Delta^s_+W^r_p$, which implies

$$\begin{aligned}d_m(\Delta^s_+W^r_p,\Delta^s_+L_q)_{L_q} &\ge d_m(S^+_p(\Psi^N_{p,r,s,k}),\Delta^s_+L_q)_{L_q}\\ &\ge 2^{-1+1/q}d_m(S^+_1(\Psi^N_{p,r,s,k}),\Delta^s_+L_1)_{L_1},\end{aligned} \tag{4.4}$$

where in the second inequality we used the facts that $S^+_p(\Psi^N_{p,r,s,k})\subseteq S^+_1(\Psi^N_{p,r,s,k})$, $\Delta^s_+L_q\subseteq\Delta^s_+L_1$ and $\|x\|_{L_q}\ge 2^{-1+1/q}\|x\|_{L_1}$.

Fix some $\epsilon>0$ to be prescribed and let $M^m_\epsilon\subset L_1$, of dimension $m<N-2$, be such that

$$d_m(S^+_1(\Psi^N_{p,r,s,k}),\Delta^s_+L_1)_{L_1} \ge E(S^+_1(\Psi^N_{p,r,s,k}),M^m_\epsilon\cap\Delta^s_+L_1)_{L_1}-\epsilon. \tag{4.5}$$

If $L^{m+1}_\epsilon\supset M^m_\epsilon$ is a subspace of L_1 of $\dim L^{m+1}_\epsilon=m+1$, then it follows that

$$d_m(S^+_1(\Psi^N_{p,r,s,k}),\Delta^s_+L_1)_{L_1} \ge E(S^+_1(\Psi^N_{p,r,s,k}),L^{m+1}_\epsilon\cap\Delta^s_+L_1)_{L_1}-\epsilon.$$

We take $\xi_{\epsilon,i} \in L_\epsilon^{m+1} \cap \Delta_+^s L_1$, $i = 1, \dots, N$, such that

$$\max_{1 \le i \le N} \|\psi_{p,r,s,k,N,i} - \xi_{\epsilon,i}\|_{L_1} \le E(S_1^+(\Psi_{p,r,s,k}^N), L_\epsilon^{m+1} \cap \Delta_+^s L_1)_{L_1} + \epsilon,$$

and extend them by $\xi_{\epsilon,i}(t) = 0$, $t \in \mathbb{R} \setminus I$, $i = 1, \dots, N$, in order to define the Steklov mean

$$\xi_{s,\eta,\epsilon,i}(t) := \eta^{-s-1} \int_0^\eta \cdots \int_0^\eta \xi_{\epsilon,i}(t + t_1 + \cdots + t_{s+1}) dt_{s+1} \cdots dt_1, \quad t \in \mathbb{R}, \quad i = 1, \dots, N.$$

It is well known that $\xi_{s,\eta,\epsilon,i} \in C^s(\mathbb{R})$, and

$$\xi_{s,\eta,\epsilon,i}^{(s)}(t) = \eta^{-s-1} \int_0^\eta \Delta_\eta^s \xi_{\epsilon,i}(t+\tau) d\tau, \quad t \in [-7/8, 7/8], \quad i = 1, \dots, N.$$

We conclude that $\xi_{s,\eta,\epsilon,i}^{(s)}$, $1 \le i \le N$, is continuous and nonnegative in $[-\frac{7}{8}, \frac{7}{8}]$, and $\xi_{s,\eta,\epsilon,i}^{(s-1)}$, $1 \le i \le N$, is continuous and nondecreasing there. Also

$$\lim_{\eta \to +0} \|\xi_{\epsilon,i} - \xi_{s,\eta,\epsilon,i}\|_{L_1(\mathbb{R})} = 0, \quad i = 1, \dots, N.$$

Thus we fix $0 < \eta \le (8(s+1))^{-1}$, so that

$$\max_{1 \le i \le N} \|\xi_{\epsilon,i} - \xi_{s,\eta,\epsilon,i}\|_{L_1(\mathbb{R})} \le \epsilon,$$

and it follows that

$$\max_{1 \le i \le N} \|\psi_{p,r,s,k,N,i} - \xi_{s,\eta,\epsilon,i}\|_{L_1} \le E(S_1^+(\Psi_{p,r,s,k}^N), L_\epsilon^{m+1} \cap \Delta_+^s L_1)_{L_1} + 2\epsilon. \tag{4.6}$$

Let $\{\zeta_{\epsilon,j}\}_{j=1}^{m+1}$ be a basis of L_ϵ^{m+1} and extend $\zeta_{\epsilon,j}(t) = 0$, $t \in \mathbb{R} \setminus I$, $j = 1, \dots, m+1$. Again, let

$$\zeta_{s,\eta,\epsilon,j}(t) := \eta^{-s-1} \int_0^\eta \cdots \int_0^\eta \zeta_{\epsilon,j}(t + t_1 + \cdots + t_{s+1}) dt_{s+1} \cdots dt_1, \quad t \in \mathbb{R}, \quad j = 1, \dots, m+1,$$

and denote their span by $L_{s,\eta,\epsilon}^{m+1}$. Also, let $\Delta_+ D^{s-1} L_{s,\eta,\epsilon}^{m+1}$ denote the set of all elements $\xi \in L_{s,\eta,\epsilon}^{m+1}$ such that $\xi^{(s-1)}$ is continuous and nondecreasing in $[-\frac{7}{8}, \frac{7}{8}]$. It follows from the above that $\xi_{s,\eta,\epsilon,i} \in \Delta_+ D^{s-1} L_{s,\eta,\epsilon}^{m+1}$. Therefore (4.6) implies

$$\begin{aligned} &E(S_1^+(\Psi_{p,r,s,k}^N), L_\epsilon^{m+1} \cap \Delta_+^s L_1)_{L_1} \\ &\ge E(S_1^+(\Psi_{p,r,s,k}^N), \Delta_+ D^{s-1} L_{s,\eta,\epsilon}^{m+1})_{L_1[-\frac{7}{8},\frac{7}{8}]} - 2\epsilon, \end{aligned} \tag{4.7}$$

and we will show that for an appropriate k,

$$E(S_1^+(\Psi_{p,r,s,k}^N),\Delta_+D^{s-1}L_{s,\eta,\epsilon}^{m+1})_{L_1[-\frac{7}{8},\frac{7}{8}]} \geq (2^{s^2+s}-2^{s-9})c(p,r,s)(kN)^{-r+s+1/p-3}. \tag{4.8}$$

Indeed, by virtue of (4.1) and (4.3) we obtain, for each $i=1,\dots,N$,

$$\begin{aligned}\psi_{p,r,s,k,N,i}^{(s-1)}(t) &:= \int_{-1}^{t}\phi_{p,r-s,4kN,4k(i-1)+1}(t_s)dt_s\\ &= c(p,r,s)(kN)^{-r+s+1/p-1}, \quad t\geq \tau_{kN,k(i-1)+1},\end{aligned}$$

where

$$c(p,r,s) := \begin{cases} 2^{1-1/p}, & s=r,\\ 2^{-2r+2s+2-1/p}, & s<r.\end{cases}$$

Set

$$\chi_{p,r,s,k,N,i}(t) = \frac{1}{(s-1)!}c(p,r,s)(kN)^{-r+s+1/p-1}(t-\bar\tau_{kN,k(i-1)+1})_+^{s-1}.$$

Then

$$\chi_{p,r,s,k,N,i}^{(s-1)}(t) = \psi_{p,r,s,k,N,i}^{(s-1)}(t), \quad t<\tau_{kN,k(i-1)} \quad \text{and} \quad t>\tau_{kN,k(i-1)+1},$$

while for $t\in(\bar\tau_{kN,k(i-1)+1},\tau_{kN,k(i-1)+1}]$, it follows from (4.1) that

$$\chi_{p,r,s,k,N,i}^{(s-1)}(t)-\psi_{p,r,s,k,N,i}^{(s-1)}(t)\leq c(p,r,s)(kN)^{-r+s+1/p}(\tau_{kN,k(i-1)+1}-t).$$

Finally, by the symmetry of $\phi_{p,r-s,kN,k(i-1)+1}$ about $\bar\tau_{kN,k(i-1)+1}$, it is readily seen that

$$\begin{aligned}&\chi_{p,r,s,k,N,i}^{(s-1)}(t-\bar\tau_{kN,k(i-1)+1})-\psi_{p,r,s,k,N,i}^{(s-1)}(t-\bar\tau_{kN,k(i-1)+1})\\ &= -(\chi_{p,r,s,k,N,i}^{(s-1)}(-t+\bar\tau_{kN,k(i-1)+1})-\psi_{p,r,s,k,N,i}^{(s-1)}(-t+\bar\tau_{kN,k(i-1)+1})).\end{aligned}$$

The last two relations imply that $\chi_{p,r,s,k,N,i}^{(s-2)}(t)-\psi_{p,r,s,k,N,i}^{(s-2)}(t)=0$, if $t<\tau_{kN,k(i-1)}$ and if $t>\tau_{kN,k(i-1)+1}$, and this in turn yields

$$\|\psi_{p,r,s,k,N,i}^{(s-3)}-\chi_{p,r,s,k,N,i}^{(s-3)}\|_{L_\infty(\mathbb{R})}\leq \frac{1}{3}2^{-6}c(p,r,s)(kN)^{-r+s+1/p-3}.$$

Hence, we have for $i=1,\dots,N$,

$$\begin{aligned}\|\psi_{p,r,s,k,N,i}-\chi_{p,r,s,k,N,i}\|_{L_1[-\frac{7}{8},\frac{7}{8}]} &\leq \left(\frac{7}{4}\right)^{s-2}\frac{1}{3}2^{-6}c(p,r,s)(kN)^{-r+s+1/p-3}\\ &\leq 2^{s-9}c(p,r,s)(kN)^{-r+s+1/p-3}.\end{aligned}$$

If $\chi^N_{p,r,s,k} := \{\chi_{p,r,s,k,N,i}\}_{i=1}^N$, then the above implies that

$$\begin{aligned}&E(S_1^+(\Psi^N_{p,r,s,k}),\Delta_+D^{s-1}L^{m+1}_{s,\eta,\epsilon})_{L_1[-\frac78,\frac78]}\\&\geq E(S_1^+(\chi^N_{p,r,s,k}),\Delta_+D^{s-1}L^{m+1}_{s,\eta,\epsilon})_{L_1[-\frac78,\frac78]}-2^{s-9}c(p,r,s)(kN)^{-r+s+1/p-3}.\end{aligned}$$

Thus (4.8) follows if we show that for an appropriate k,

$$E(S_1^+(\chi^N_{p,r,s,k}),\Delta_+D^{s-1}L^{m+1}_{s,\eta,\epsilon})_{L_1[-\frac78,\frac78]}\geq 2^{s^2+s}c(p,r,s)(kN)^{-r+s+1/p-3}. \tag{4.9}$$

To this end, we first prove that

$$\begin{aligned}&E(d^{s-1}S_1^+(\chi^N_{p,r,s,k}),d^{s-1}\Delta_+D^{s-1}L^{m+1}_{s,\eta,\epsilon})_{L_1[-\frac14,\frac14]}\\&\geq 2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2},\end{aligned}\tag{4.10}$$

where for a subset $U\subset X$ we use the notation

$$d^{s-1}U:=\{x^{(s-1)}\mid x\in U\}.$$

Indeed, let $I_{k,N,i}:=[\bar\tau_{kN,k(i-1)+1},\tau_{kN,ki}]$, $i=1,\ldots,N$ and define the discretization operator $A_{k,N}: L_1\ni x\to A_{k,N}x\in l_1^N$ by

$$A_{k,N}x:=\left(\int_{I_{k,N,1}}x(t)dt,\ldots,\int_{I_{k,N,N}}x(t)dt\right).$$

Then it is easy to see that

$$\|A_{k,N}x\|_{l_1^N}\leq\|x\|_{L_1[-\frac14,\frac14]}.$$

Hence

$$\begin{aligned}&E(d^{s-1}S_1^+(\chi^N_{p,r,s,k}),d^{s-1}\Delta_+D^{s-1}L^{m+1}_{s,\eta,\epsilon})_{L_1[-\frac14,\frac14]}\\&\geq E(A_{k,N}d^{s-1}S_1^+(\chi^N_{p,r,s,k}),A_{k,N}d^{s-1}\Delta_+D^{s-1}L^{m+1}_{s,\eta,\epsilon})_{l_1^N}\\&\geq d_{m+1}(A_{k,N}S_1^+(D^{s-1}\chi^N_{p,r,s,k}),\Delta^1_+)_{l_1^N},\end{aligned}\tag{4.11}$$

since

$$A_{k,N}d^{s-1}\Delta_+D^{s-1}L^{m+1}_{s,\eta,\epsilon}\subseteq\Delta^1_+\subset\mathbb{R}^N,$$

where Δ^1_+ is defined in Lemma D with n replaced by N.

Now

$$A_{k,N}\chi^{(s-1)}_{p,r,s,k,N,i}=2^{-1}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2}\tilde e^{(i)},\quad i=1,\ldots,N,$$

where $\tilde e^{(i)}$ are the N-tuples from (4.4) (with n replaced by N). Hence

$$A_{k,N}d^{s-1}S_1^+(\chi^N_{p,r,s,k})=2^{-1}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2}S_1^+(\tilde E^N),$$

where $\tilde{E}^N := \{\tilde{e}^{(i)}\}_{i=1}^N$. Therefore

$$\begin{aligned} &d_{m+1}(A_{k,N}d^{s-1}S_1^+(\chi_{p,r,s,k}^N),\Delta_+^1)_{l_1^N} \\ &= 2^{-1}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2}d_{m+1}(S_1^+(\tilde{E}^N)),\Delta_+^1)_{l_1^N}. \end{aligned} \tag{4.12}$$

For $m < N-2$, it follows by Lemma D that

$$d_{m+1}(S_1^+(\tilde{E}^N)),\Delta_+^1)_{l_1^N} \geq 1/8,$$

and combining with (4.11) we obtain (4.10).

Now, by (4.10) there exists an $1 \leq i_0 \leq N$ so that

$$\|\chi_{p,r,s,k,N,i_0}^{(s-1)} - \xi^{(s-1)}\|_{L_1[-\frac{1}{4},\frac{1}{4}]} \geq 2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2},$$

for all $\xi \in \Delta_+ D^{s-1}L_{s,\eta,\epsilon}^{m+1}$. Denote $I_{i_0} := [-\frac{1}{2} + \bar{\tau}_{kN,k(i_0-1)+1}, \bar{\tau}_{kN,k(i_0-1)+1} + \frac{1}{2}]$. Then for each $\xi \in \Delta_+ D^{s-1}L_{s,\eta,\epsilon}^{m+1}$,

$$\|\chi_{p,r,s,k,N,i_0}^{(s-1)} - \xi^{(s-1)}\|_{L_1(I_{i_0})} \geq 2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2}. \tag{4.13}$$

Recall that $\xi^{(s-1)}$ is continuous and nondecreasing in $[-\frac{7}{8},\frac{7}{8}]$, thus in $I_{i_0} \subseteq [-\frac{3}{4},\frac{3}{4}]$. So if

$$0 \leq \xi^{(s-1)}(t) \leq c(p,r,s)(kN)^{-r+s+1/p-1}, \quad t \in I_{i_0}, \tag{4.14}$$

then we may apply Corollary 1 (with s replaced by $s-1$) with $\tau = \bar{\tau}_{kN,k(i_0-1)+1}$, $b = c(p,r,s)(kN)^{-r+s+1/p-1}$, $a = \frac{1}{2}$, and $A = 2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2}$, to obtain

$$\|\chi_{p,r,s,k,N,i_0} - \xi\|_{L_1(I_{i_0})} \geq 2^{-s^2-5s-10}(k-1)^2c(p,r,s)(kN)^{-r+s+1/p-3}.$$

We conclude that

$$\|\chi_{p,r,s,k,N,i_0} - \xi\|_{L_1[-\frac{7}{8},\frac{7}{8}]} \geq 2^{-s^2-5s-10}(k-1)^2c(p,r,s)(kN)^{-r+s+1/p-3}. \tag{4.15}$$

If (4.14) does not hold, then we may have that

$$\xi^{(s-1)}(-\frac{1}{2} + \bar{\tau}_{kN,k(i_0-1)+1}) \leq -2^{-1}CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1},$$

or

$$\xi^{(s-1)}(\bar{\tau}_{kN,k(i_0-1)+1} + 1/2) \geq (1 + 2^{-1}CN^{-2})c(p,r,s)(kN)^{-r+s+1/p-1}$$

for some $C > 0$, to be prescribed. In view of the monotonicity of $\xi^{(s-1)}$ in $[-\frac{7}{8},\frac{7}{8}]$, we have

$$|\chi_{p,r,s,k,N,i_0}^{(s-1)}(t) - \xi^{(s-1)}(t)| \geq 2^{-1}CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1}, \quad t \in J_0,$$

where $J_0 = [-\frac{7}{8}, -\frac{3}{4}]$ in the former case and $J_0 = [\frac{3}{4}, \frac{7}{8}]$ in the latter. Lemma 1 then implies the existence of a $J_{s-1} \subset J_0$, of length $|J_{s-1}| \geq 2^{-2s-1}$, such that

$$|\chi_{p,r,s,k,N,i_0}(t) - \xi(t)| \geq 2^{-s^2-2s-1}CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1} \quad t \in J_{s-1},$$

which in turn yields

$$\|\chi_{p,r,s,k,N,i_0} - \xi\|_{L_1(J_{s-1})} \geq 2^{-s^2-4s-2}CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1}.$$

We conclude that in either case

$$\|\chi_{p,r,s,k,N,i_0} - \xi\|_{L_1[-\frac{7}{8},\frac{7}{8}]} \geq 2^{-s^2-4s-2}Ck^2c(p,r,s)(kN)^{-r+s+1/p-3}. \tag{4.16}$$

Otherwise, again due to the monotonicity of $\xi^{(s-1)}$ in $[-\frac{7}{8}, \frac{7}{8}]$, we have

$$\begin{aligned} &-2^{-1}CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1} < \xi^{(s-1)}(t) \\ &< (1+2^{-1}CN^{-2})c(p,r,s)(kN)^{-r+s+1/p-1}, \quad t \in I_{i_0}. \end{aligned} \tag{4.17}$$

Denote

$$\zeta_{s-1}(t) := (1+CN^{-2})^{-1}(\xi^{(s-1)}(t) + 2^{-1}CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1}),$$

and it readily follows by (4.17) that $0 < \zeta_{s-1}(t) < c(p,r,s)(kN)^{-r+s+1/p-1}$, $t \in I_{i_0}$. Also, by virtue of (4.13),

$$\begin{aligned} &\|\chi^{(s-1)}_{p,r,s,k,N,i_0} - \zeta_{s-1}\|_{L_1(I_{i_0})} \\ \geq&\|\chi^{(s-1)}_{p,r,s,k,N,i_0} - \xi^{(s-1)}\|_{L_1(I_{i_0})} - \|\xi^{(s-1)} - \zeta_{s-1}\|_{L_1(I_{i_0})} \\ \geq&2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2} - \|\xi^{(s-1)} - \zeta_{s-1}\|_{C(I_{i_0})}|I_0| \\ \geq&2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2} \\ &-(1+CN^{-2})^{-1}CN^{-2}(\|\xi^{(s-1)}\|_{C(I_{i_0})} + 2^{-1}c(p,r,s)(kN)^{-r+s+1/p-1}) \\ \geq&2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2} \\ &-(1+CN^{-2})^{-1}CN^{-2}(1+2^{-1}CN^{-2}+2^{-1})c(p,r,s)(kN)^{-r+s+1/p-1} \\ \geq&2^{-4}(k-1)c(p,r,s)(kN)^{-r+s+1/p-2} - 2CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1} \\ =&(2^{-4}k^{-1}(k-1) - 2CN^{-1})c(p,r,s)(kN)^{-r+s+1/p-1}N^{-1}. \end{aligned} \tag{4.18}$$

(Observe that $N \geq 3$ and $k \geq 2$, so taking $C < 2^{-5}$ guarantees that the last quantity is positive.) Set

$$\begin{aligned} \zeta(t) :=& \sum_{\nu=0}^{s-2} \xi^{(\nu)}(\bar{\tau}_{kN,k(i_0-1)+1})(t - \bar{\tau}_{kN,k(i_0-1)+1})^\nu \\ &+ \frac{1}{(s-2)!}\int_{\bar{\tau}_{kN,k(i_0-1)+1}}^t \zeta_{s-1}(\tau)(t-\tau)^{s-2}d\tau. \end{aligned}$$

Then we note that $\zeta^{(s-1)} = \zeta_{s-1}$ is nondecreasing in I_{i_0}, so applying Corollary 1 (with s replaced by $s-1$) for $\tau = \bar{\tau}_{kN,k(i_0-1)+1}$, $b = c(p,r,s)(kN)^{-r+s+1/p-1}$, $A = (2^{-4}k^{-1}(k-1) - 2CN^{-1})c(p,r,s)(kN)^{-r+s+1/p-1}N^{-1}$ and $a = \frac{1}{2}$ we obtain

(4.19)
$$\|\chi_{p,r,s,k,N,i_0} - \zeta\|_{L_1(I_{i_0})} \geq 2^{-s^2-5s-10}(k-1-2^5CkN^{-1})^2 c(p,r,s)(kN)^{-r+s+1/p-3}.$$

Now, by Taylor's formula we have

$$\zeta(t) - \xi(t) = \frac{1}{(s-2)!}\int_{\bar{\tau}_{kN,k(i_0-1)+1}}^{t} (\zeta_{s-1}(\tau) - \xi^{(s-1)}(\tau))(t-\tau)^{s-2}d\tau, \quad t \in I_{i_0}.$$

Hence, as in (4.18),

$$\begin{aligned}\|\zeta - \xi\|_{L_1(I_{i_0})} &\leq \frac{2^{-s+1}}{s!}\|\zeta_{s-1} - \xi^{(s-1)}\|_{C(I_{i_0})} \\ &\leq \frac{2^{-s+2}}{s!}CN^{-2}c(p,r,s)(kN)^{-r+s+1/p-1} \\ &= \frac{2^{-s+2}}{s!}Ck^2c(p,r,s)(kN)^{-r+s+1/p-3},\end{aligned}$$

which by (4.19) yields

$$\begin{aligned}&\|\chi_{p,r,s,k,N,i_0} - \xi\|_{L_1(I_{i_0})} \\ &\geq \|\chi_{p,r,s,k,N,i_0} - \zeta\|_{L_1(I_{i_0})} - \|\zeta - \xi\|_{L_1(I_{i_0})} \\ &\geq 2^{-s^2-5s-10}((k-1-2^5CkN^{-1})^2 - 2^{s^2+4s+12}Ck^2)c(p,r,s)(kN)^{-r+s+1/p-3}.\end{aligned}$$

So we conclude that

(4.20)
$$\begin{aligned}&\|\chi_{p,r,s,k,N,i_0} - \xi\|_{L_1[-\frac{7}{8},\frac{7}{8}]} \\ &\geq 2^{-s^2-5s-10}((k-1-2^5CkN^{-1})^2 - 2^{s^2+4s+12}Ck^2)c(p,r,s)(kN)^{-r+s+1/p-3}.\end{aligned}$$

If we take

$$\begin{aligned}c(s,k,C) := \min\{&2^{-s^2-5s-10}(k-1)^2, 2^{-s^2-4s-2}Ck^2, \\ &2^{-s^2-5s-10}((k-1-2^5CkN^{-1})^2 - 2^{s^2+4s+12}Ck^2)\}\end{aligned}$$

then by combining (4.15), (4.16) and (4.20) we have, for each $\xi \in \Delta_+ D^{s-1}L^{m+1}_{s,\eta,\epsilon}$,

$$\|\chi_{p,r,s,k,N,i_0} - \xi\|_{L_1[-\frac{7}{8},\frac{7}{8}]} \geq c(s,k,C)c(p,r,s)(kN)^{-r+s+1/p-3}.$$

Now, straightforward computations show that if we take $C := 2^{-s^2-4s-14}$, then $c(s,k,C) \geq 2^{-2}k^2$, so that taking $k = 2^{s^2+3s+6}$ yields (4.9), and in turn proves

(4.8). By virtue of (4.4) through (4.7) we conclude that

$$\begin{aligned}&d_m(\Delta_+^s W_p^r, \Delta_+^s L_q)_{L_q}\\ &\geq 2^{-1+1/q}(2^{s^2+s} - 2^{s-9})2^{(s^2+3s+6)(-r+s+1/p-3)}c(p,r,s)N^{-r+s+1/p-3} - 3\epsilon,\end{aligned}$$

for every $\epsilon > 0$, whence without ϵ too, and for $m = n$, $N = n+3$, we obtain

$$d_n(\Delta_+^s W_p^r, \Delta_+^s L_q)_{L_q} \geq cn^{-r+s+1/p-3},$$

where $c = c(r,s,p,q) > 0$. This completes the proof of the lower bounds in (1.1) for $s = 3,\ldots,r$.

For $s = r+1$, we observe that the lower bounds in (1.2) are independent of $1 \leq p \leq \infty$, so it suffices to establish them for the smallest class, namely, for $\Delta_+^{r+1}W_\infty^r$, since $\Delta_+^{r+1}W_\infty^r \subseteq \Delta_+^{r+1}W_p^r$, $1 \leq p \leq \infty$. We also note that

$$\chi_{\infty,r,k,N,i}(t) = \frac{1}{r!}(t - \bar{\tau}_{kN,k(i-1)+1})^r, \quad i = 1,\ldots,N$$

is differentiable r times and $\chi_{\infty,r,k,N,i} \in \Delta_+^{r+1}W_\infty^r$, $1 \leq i \leq N$. Thus we do not need the elaborate construction we had before and can work directly with $\chi_{\infty,r,k,N,i}$, $1 \leq i \leq N$. Therefore, if we denote $\chi_{\infty,r,k}^N := \{\chi_{\infty,r,k,N,i}\}_{i=1}^N$, then $S_1^+(\chi_{\infty,r,k}^N) \subset \Delta_+^{r+1}W_\infty^r$. Using the discretization operator $A_{k,N}$, defined above, we prove as before (see (4.12)) that

$$\begin{aligned}d_{m+1}(A_{k,N}d^r S_1^+(\chi_{\infty,r,k}^N), \Delta_+^1)_{l_1^N} &\geq 2^{-1}(k-1)kN^{-1}d_{m+1}(S_1^+(\tilde{E}^N)), \Delta_+^1)_{l_1^N}\\ &\geq 2^{-4}(k-1)kN^{-1}.\end{aligned}$$

Then, we proceed as before to conclude that

$$d_n(\Delta_+^{r+1}W_\infty^r, \Delta_+^1 L_q)_{L_q} \geq cn^{-2},$$

where $c = c(r,q) > 0$. This completes the proof of the lower bound in (1.2), and concludes the proof of our theorem. ∎

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